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PARACOMPACTNESS AND PRODUCTS

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We investigate paracompactness in the product of a paracompact space Y with a paracompact linearly ordered topological space X, where certain controls are placed on the subspace ηX of points without a compact neighborhood. If ηX is dispersed, then $X^\alpha \times Y$ is paracompact for all finite ordinals α. If the Lindelöf degree of Y is less than the least cardinal number, for each $n \in \eta X$, of any collection of neighborhoods whose intersection does not contain n in its interior, then $X \times Y$ is paracompact.

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Many of the standard examples showing that the product of two paracompact spaces is not always paracompact require one or both factors to be a linearly ordered topological space (LOTS) or a subspace of a LOTS. We illustrate some sufficient conditions for the product of two paracompact spaces to be paracompact when one factor is a LOTS (or a subspace of a LOTS). All spaces are assumed Hausdorff.

1. Notation

A *linearly ordered topological space* (LOTS) is a linearly ordered set with its interval topology. An *interior gap* of a LOTS X is a Dedekind cut $(A|B)$ of X such that A has no supremum (sup) and B has no infimum (inf). An *end-gap*, left or right, means the absence of an infimum (inf) or supremum (sup) of the linearly ordered set. The *Dedekind compactification* X^+ of a LOTS X is formed by suitably ordering $X \cup \{g: g$ is a gap of $X\}$ in a manner similar to the completion of the rationals. X is dense in the compact LOTS X^+. A gap is a *Q-gap* provided it is the limit of an increasing sequence, if not a right end-gap, and a decreasing sequence, if not a left end-gap, each having no limit points in X. For further details on LOTS, their gaps, and their compactifications, we suggest [4].

If X is any linearly ordered set, then X^* denotes the same set with the inverse order of X.

Intervals in a LOTS X are denoted by $[a, b]$ when closed and by $]a, b[$ when open, and in the latter case a and/or b may be allowed to be a gap. A convex set C satisfies $a, b \in C$ and $a < x < b$ implies $x \in C$. Singleton sets are considered to be convex.

Let X be a LOTS, for $x \in X^+$ let $\tau_+(x)$ (resp. $\tau_-(x)$) be the smallest ordinal λ for which there is a decreasing (resp. increasing) sequence of type λ in X^+ coinitial (resp. cofinal) with x. Then the *type* of x is $\tau(x) = \inf\{\tau_+(x), \tau_-(x)\}$ while the character of x is $\chi(x) = \sup\{\tau_+(x), \tau_-(x)\}$.

The *Lindelöf degree* $L(X)$ of a topological space is the smallest cardinal α such that each open covering of X has a subcovering of cardinal less than or equal to α.

For a topological space X, we define ηX to be the subspace of all points of X without a compact neighborhood.

A topological space is called *dispersed*[1] if every non-empty subspace has an isolated point.

In this paper we make a slight deviation from the usual definition for the term "refinement" in that a refinement of a covering need not be a covering of any set unless it is so stated.

2. Remarks and examples

A list of papers concerning the normality or paracompactness of products would be long, and hence we shall not present one here; however, certain of these results should be mentioned for their strong relationship with our work.

2.1. Katuta [6] has shown that if X is paracompact, and $X \times Y$ is normal for every paracompact space Y, then $X \times Y$ is paracompact.

[1] Dispersed spaces are also called scattered spaces in the literature.

2.2. The disjoint union of the Michael line [8] and the Sorgenfrey line [10] can be embedded as a closed subspace of a first-countable hereditarily paracompact LOTS L using a technique of Fedorčuk [3] or Lutzer (see discussion after Corollary 3.3). Hence $L \times L$ and $L \times P$ are not normal, where P is the space of irrationals.

2.3. Przymusinski [9], armed with Martin's Axiom and $2^{\aleph_0} = 2^{\aleph_1}$, has given the Sorgenfrey topology to a subset X of $\mathbf{R}$ to produce a hereditarily paracompact space whose product with itself is perfectly normal and not paracompact.

2.4. That a separable LOTS is paracompact follows from results in [4]. That a separable LOTS is also an M-space[2] is known as a "folk result" to the author [13]. It is well-known that the product of two paracompact M-spaces is paracompact.

2.5. Each of the spaces L in 2.2 and X in 2.3 is a paracompact non-M-space. Consider the space Y to be the long line [11] with the last point added, the countable ordinals subtracted, and given the interval topology. Y is a paracompact non-M-space; however, Theorem 3.3 will show that $Y \times Z$ is paracompact for any Lindelöf space Z. Further, Theorem 3.4 will show that $Y \times Z$ is paracompact for any paracompact space Z. On the other hand, the results in [12] do not seem to be applicable in this example.

2.6. For results concerning normality in products of spaces, we suggest [1], [2] and [5].

3. Theorems

As a trivial corollary of a result of Suzuki [12] we obtain a very necessary lemma. Each of our theorems is a generalization of this result.

3.1. **Lemma.** *If X and Y are paracompact and X is a locally compact LOTS, then $X \times Y$ is paracompact.*

[2] X is an M-space if there is a closed continuous function f from X onto a metric space Y such that $f^{-1}(y)$ is countably compact for each $y \in Y$.

3.2. Lemma [4]. *A LOTS X is paracompact if and only if every gap of X is a Q-gap.*

3.3. Theorem. *Let X and Y be paracompact spaces, and let X be a LOTS for which $\tau(n) > L(Y)$ or $\tau(n) = 0$ and $\chi(n) > L(Y)$ for each $n \in \eta X$. Then $X \times Y$ is paracompact.*

Proof. Let R be an open covering of $X \times Y$ and $A \subseteq X$. For convenience we say that S is a *pro-cover of A* provided S is an open σ-locally-finite refinement of R covering $A \times Y$.

For $x_1, x_2 \in X$ define $x_1 \sim x_2$ if and only if there is a pro-cover of

$$[\inf\{x_1, x_2\}, \sup\{x_1, x_2\}] .$$

Then $\sim$ is an equivalence relation on X whose classes we designate $\mathrm{kl}(x)$. It will be shown that x is an interior point of $\mathrm{kl}(x)$ and there is a pro-cover of $\mathrm{kl}(x)$ for each $x \in X$. From this it follows that $X \times Y$ is paracompact.

x is an interior point of $\mathrm{kl}(x)$. Indeed, it is obvious from Lemma 3.1 if $x \notin \eta X$. On the other hand, if $x \in \eta X$, there is a refinement R_1 of R covering $\{x\} \times Y$ such that R_1 consists entirely of members each of the form $C \times U$, where C is open and convex in X and U is open in Y. Choose an open locally-finite refinement H of $\{U: C \times U \in R_1\}$ covering Y, and for each $W \in H$ choose $C(W)$ to be open and convex in X and $U(W)$ open in Y such that $W \subseteq U(W)$ and $C(W) \times U(W) \in R_1$. Then

$$R_2 = \{C(W) \times W: W \in H\}$$

is a pro-cover of $\{x\}$. Let $R_3 \subseteq R_2$ be a covering of $\{x\} \times Y$ such that $|R_3| \leq L(Y)$.

Suppose $\tau_+(x) > 0$; then $\tau_+(x) > |R_3|$, so

$$x < \inf\{\sup C(W): C(W) \times W \in R_3\}.$$

Similarly, $\tau_-(x) > 0$ implies

$$x > \sup\{\inf C(W): C(W) \times W \in R_3\} .$$

In any case, x is in the interior of

$$\bigcap\{C(W) : C(W) \times W \in R_3\} ,$$

and R_3 is a pro-cover of the latter. Therefore, x is an interior point of $\mathrm{kl}(x)$; therefore, $\mathrm{kl}(x)$ is an open and closed convex set in X.

Suppose $\inf(\mathrm{kl}(x)) \in X$ and $\sup(\mathrm{kl}(x)) = g \notin X$, then $\inf(\mathrm{kl}(x))$ is in the interior of $\mathrm{kl}(\inf(\mathrm{kl}(x)))$, from which it follows that $\inf(\mathrm{kl}(x)) \in \mathrm{kl}(x)$ and $\inf(\mathrm{kl}(x)) = \inf X$. So there is a pro-cover T of $[\inf X, x]$. On the other hand, from Lemma 3.2, there is an increasing sequence

$$\{x_\beta : \beta < \tau_-(g)\} \subseteq X^+$$

cofinal with g such that

(1) $x_0 = x$,

(2) if $\tau_-(g) = \omega$, we may choose $x_\beta \in X$ for all β ,

(3) $\{x_\beta : \beta < \tau_-(g)\}$ has no limit points in X.

If $\tau_-(g) = \omega$, we choose S_β to be a pro-cover of $[x_0, x_{\beta+1}]$ for all $\beta < \omega$. In this case,

$$T \cup \bigcup\{S_\beta : \beta < \omega\}$$

is a pro-cover of $\mathrm{kl}(x)$.

If $\tau_-(g) > \omega$, we choose S_0 to be a pro-cover of $[x_0, x_1[$ and S_β to be a pro-cover of

$$]\sup\{x_\gamma : \gamma < \beta\}, x_{\beta+1}[$$

for all $\beta > 0$ such that

$$\bigcup\{V : V \in S_\beta\} =]\sup\{x_\gamma : \gamma < \beta\}, x_{\beta+1}[\times Y.$$

It is clear that each of the following is a σ-locally-finite collection:

(a) $(S_0 \cup T) \cap ([\inf X, x_1[\times Y)$;

(b) $\bigcup\{S_\lambda : \lambda \text{ is a limit ordinal}\}$;

(c) $\{V : V \in S_{\lambda+k}, \lambda \text{ is a limit ordinal}, 0 < \lambda < \tau_-(g)\}$ for $0 < k < \omega$.

Therefore, the union of these collections is a pro-cover of $\mathrm{kl}(x)$.

There are three other cases to examine: $\inf(\mathrm{kl}(x)) \in X$ and $\sup(\mathrm{kl}(x)) \in X$; $\inf(\mathrm{kl}(x)) \notin X$ and $\sup(\mathrm{kl}(x)) \in X$; $\inf(\mathrm{kl}(x)) \notin X$ and $\sup(\mathrm{kl}(x)) \notin X$. However, in each case a pro-cover of $\mathrm{kl}(x)$ can be constructed with a modification of the techniques of the previous paragraphs.

Since $\mathrm{kl}(x)$ is clopen for each $x \in X$ and $\mathrm{kl}(x)$ has a pro-cover P_x for each $x \in X$ such that

$$\cup P_x = \mathrm{kl}(x) \times Y,$$

it follows that X has a pro-cover. □

If, in Theorem 3.3, Y is in fact a LOTS, then $\chi(y) \leq L(Y)$ for each $y \in Y^+ - Y$. We conjecture that substitution of "$\tau(n) > \chi(y)$ or $\tau(n) = 0$ and $\chi(n) > \chi(y)$ for each $n \in \eta X$ and $y \in Y^+ - Y$" is not sufficient to reach $X \times Y$ paracompact.

3.4. Theorem. *Let X and Y be paracompact, and let X be a LOTS for which ηX is dispersed. Then $X \times Y$ is paracompact.*

Proof. For each pair $x_1, x_2 \in X$ we say $x_1 \sim x_2$ if and only if

$$[\inf\{x_1, x_2\}, \sup\{x_1, x_2\}] \times Y$$

is paracompact. Then $\sim$ is an equivalence relation on X whose classes we denote $\mathrm{kl}(x)$. We show that $\mathrm{kl}(x) \times Y$ is paracompact and x is an interior point of $\mathrm{kl}(x)$ for each $x \in X$. From this it follows that $X \times Y$ is paracompact.

Form the derivatives of ηX by allowing $D_0 = \eta X$, D_α to be the set of non-isolated points of $D_{\alpha-1}$ when α is non-limit, and $D_\alpha = \bigcap\{D_\beta : \beta < \alpha\}$ when α is a limit ordinal. For $x \in X$ call $\alpha(x)$ the first ordinal such that $x \notin D_{\alpha(x)}$. Our proof is by induction.

Suppose $\alpha(x) = 0$; we may use Lemma 3.1 to show that x is an interior point of $\mathrm{kl}(x)$. Moreover, if $\sup(\mathrm{kl}(x)) \notin D_0$ or if $\sup(\mathrm{kl}(x))$ is not the limit of an increasing sequence of gaps, then a modification of the third paragraph of Theorem 3.3 shows $[x, \sup(\mathrm{kl}(x))] \times Y$ to be paracompact. Hence we suppose for $k = \sup(\mathrm{kl}(x))$ that $k \in D_0$ and k is the limit of an increasing sequence of gaps and R is an open cover of $[x, k] \times Y$. There exists a collection $R_1 = \{U_i : i \in I\}$ of open sets of $X \times Y$ such that

(1) each U_i is of the form $C_i \times G_i$ for each $i \in I$,
(2) C_i is a convex set with $g_i = \inf C_i \notin X$ for each $i \in I$,
(3) $\{G_i : i \in I\}$ is a locally finite covering of Y,
(4) R_1 is a locally finite refinement of R covering $\{k\} \times Y$.

For each $i \in I$ choose S_i to be an open locally finite refinement of R covering $]x, g_i[\times Y$ and let

$$T_i = \{V \cap (]x, g_i[\times G_i) : V \in S_i\}.$$

Then $\{R_1\} \cup \bigcup\{T_i: i \in I\}$ is an open locally finite refinement of R covering $]x, k] \times Y$. Repetition of these techniques shows that $[\inf(\mathrm{kl}(x)), x] \times Y$, and hence $\mathrm{kl}(x) \times Y$, is paracompact and closed in X.

Suppose $n \in \eta X$ and $\alpha(x) < \alpha(n)$ implies $\mathrm{kl}(x) \times Y$ is paracompact, x is an interior point of $\mathrm{kl}(x)$, and $\mathrm{kl}(x)$ is closed. We consider the case $\tau(n) > 0$. There exist $a, b \in X$ such that $n \in]a,b[$, $]a,n[\neq \emptyset$, $]n,b[\neq \emptyset$, $\alpha(a) < \alpha(x)$, $\alpha(b) < \alpha(x)$, and

$$[a, b] \cap D_{\alpha(n)-1} = \{n\} .$$

Whether or not $[a, b]$ is the disjoint union of clopen convex sets each of whose product with Y is paracompact, we have

$$\inf(\mathrm{kl}(b)) \leqslant n \leqslant \sup(\mathrm{kl}(a)) ,$$

since $\mathrm{kl}(c) \times Y$ is paracompact for all $c \in [a, b] - \{n\}$. Since $\mathrm{kl}(a)$ and $\mathrm{kl}(b)$ are closed, $[a, b] \times Y$ is paracompact and n is in the interior of $\mathrm{kl}(n)$. On the other hand, $n \subset \mathrm{kl}(a)$, so $\mathrm{kl}(n) \times Y$ is paracompact. The cases for $\tau(n) = 0$ are similar. □

3.5. Corollary. *If X is a paracompact LOTS and ηX is dispersed, then X^α is paracompact for all finite ordinals α.*

Lutzer [7] has shown that if X is a subspace of a LOTS, then there exists an order-preserving homeomorphism from X onto a closed subspace of a certain LOTS L obtained by replacing points which are

(1) suprema of sequences but not limit points of those sequences with ω^*,

(2) infima of sequences but not limit points of those sequences with ω,

(3) suprema and infima of sequences but are isolated points with $\omega^* + \omega$.

Lutzer has shown that L is paracompact whenever X is paracompact. It is easy to show that ηL is dispersed whenever ηX is dispersed. Hence we would have the following:

3.6. Corollary. *If X is a paracompact subspace of a LOTS and ηX is dispersed, then $X \times Y$ is paracompact for any paracompact space Y.*

Similarly, we may apply Theorem 3.3 to arrive at:

3.7. Corollary. *Let X be a paracompact subspace of a LOTS, and let Y be a topological space. If $L(Y)$ is less than the least cardinal number, for each $n \in \eta X$, of any collection of neighborhoods of n whose intersection does not contain n in its interior, then $X \times Y$ is paracompact.*

We make two conjectures sparked by recent theorems and counterexamples in the area of box topologies:[3]

3.8. Assume the continuum hypothesis. If X_n is a paracompact LOTS for which ηX_n is dispersed for each $n \in \omega$, then $\Pi\{X_n : n \in \omega\}$ is paracompact when given the box topology.

3.9. There is a paracompact space X for which ηX is dispersed and a metric space Y such that $X \times Y$ is not paracompact.

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